

Optical Propagation in Three-Layer Lossy Media Using Transfer Matrix Method

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Abstract: This paper presents a generalized transfer matrix method (TMM) for analyzing optical propagation in three-layer periodic structures composed of alternating transparent and complex refractive index media. Unlike conventional approaches, this work accounts for the decoupling of planes of constant amplitude and phase within lossy media, where the complex wave vector introduces non-orthogonal propagation characteristics. By separately applying Snell's law to the amplitude and phase unit vectors, we derive explicit expressions for the effective phase and amplitude refractive indices. The transmission and dispersion relations for both TE and TM polarizations are established via boundary continuity conditions and Bloch's theorem. Furthermore, we demonstrate that the derived transmission matrix for lossy media can be mathematically transformed into an equivalent lossless form through complex effective index and thickness renormalization. This provides a practical pathway for implementing lossy photonic crystals in commercial software such as COMSOL Multiphysics. Numerical validations confirm the accuracy and stability of the proposed method.

Keywords: Transfer matrix method; Complex refractive index; Photonic crystal; Equi-amplitude surface; COMSOL; Bloch theorem.

1. Introduction

Photonic crystals (PCs) have attracted considerable attention due to their ability to control light propagation through periodic modulation of the refractive index [1-3]. While most theoretical studies assume lossless dielectric materials, practical applications—particularly in plasmonics, absorbers, and active devices—inevitably involve materials with complex refractive indices [4,5]. The presence of an imaginary component in the refractive index introduces non-trivial modifications to wave propagation, most notably the separation of planes of constant amplitude and phase [6].

Conventional transfer matrix methods, widely used for modeling periodic structures, typically assume real refractive indices and orthogonal wave fronts. However, when lossy media are involved, the standard approach fails to correctly describe the phase accumulation and impedance matching at interfaces due to the non-orthogonality of the wave vector components [7,8]. This limitation necessitates a more rigorous formulation that explicitly treats the amplitude and phase contributions of the wave vector independently.

In this work, we develop a complete transfer matrix framework for three-layer transparent-complex-transparent periodic structures. The complex medium is characterized by a refractive index $n' = n_1 + ik$, and the wave vector is decomposed as $\vec{k} = k_s \vec{s} + ik_q \vec{q}$, where \vec{s} and \vec{q} are unit vectors along the phase and amplitude propagation directions, respectively. The angle between them, denoted as ε , is a key parameter distinguishing lossy media from lossless ones. We derive explicit expressions for the normalized phase and amplitude wave numbers N_s and N_q from the complex dispersion relation. By applying Snell's law separately to the \vec{s} and \vec{q} components, we obtain the transmitted wave vectors at each interface. The boundary conditions for electric and magnetic fields are then applied to construct the full transfer matrix for a single period. Bloch's theorem is subsequently employed to derive the complex band structure.

A significant contribution of this work is the demonstration that the transfer matrix for a lossy layer can be recast into a form identical to that of a lossless layer by introducing an effective complex refractive index and an effective complex thickness. This transformation allows the proposed method to be directly implemented in COMSOL Multiphysics without modification to the built-in electromagnetic solvers, offering a convenient and accurate simulation strategy for engineers and researchers.

The remainder of this paper is organized as follows. Section 2 details the theoretical formulation, including wave decomposition, boundary conditions, and matrix construction for both TE and TM polarizations. Section 3 presents the equivalent transformation scheme for COMSOL implementation. Section 4 provides numerical examples and validation. Section 5 concludes the work.

2. Theoretical Formulation

Consider a one-dimensional photonic crystal composed of alternating layers of non-absorbing material A (refractive index n_0) and absorbing material B (complex refractive index $n_1 + ik$), arranged periodically along the x-direction with period $\Lambda = d_a + d_b$. The refractive index profile is given by:

$$n'(x) = \begin{cases} n_0, & (0 \leq x < d_a) \\ n_1 + ik, & (d_a \leq x < \Lambda) \end{cases} \quad (1)$$

with periodic condition $n'(x + \Lambda) = n'(x)$.

In a lossless medium, the wave vector is real and perpendicular to both the planes of constant amplitude and phase, which coincide. In a lossy medium, however, the planes of constant amplitude and phase are generally not parallel. The complex wave vector is expressed as:

$$\vec{k} = k_s \vec{s} + ik_q \vec{q}$$

where \vec{s} and \vec{q} are unit vectors normal to the planes of constant amplitude and phase, respectively. The angle between them is defined as:

$$\varepsilon = \cos^{-1}(\vec{q} \cdot \vec{s})$$

Squaring the wave vector and equating to $k_0^2(n_1 + ik)^2$ yields the normalized phase and amplitude wave numbers:

$$\begin{aligned} N_s &= \frac{k_s}{k_0} = \sqrt{\frac{\sqrt{(n_1^2 - k^2)^2 + (\frac{2n_1k}{\cos\varepsilon})^2} + (n_1^2 - k^2)}{2}} \\ N_q &= \frac{k_q}{k_0} = \sqrt{\frac{\sqrt{(n_1^2 - k^2)^2 + (\frac{2n_1k}{\cos\varepsilon})^2} - (n_1^2 - k^2)}{2}} \end{aligned} \quad (2)$$

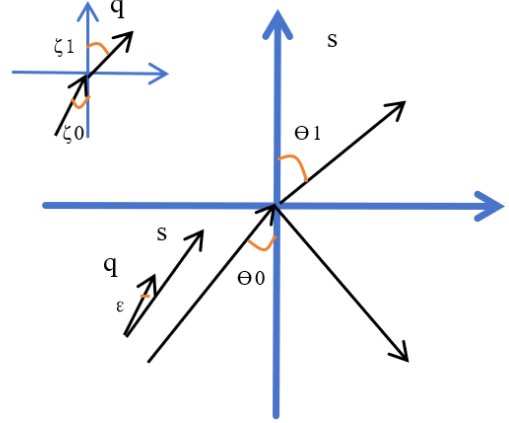
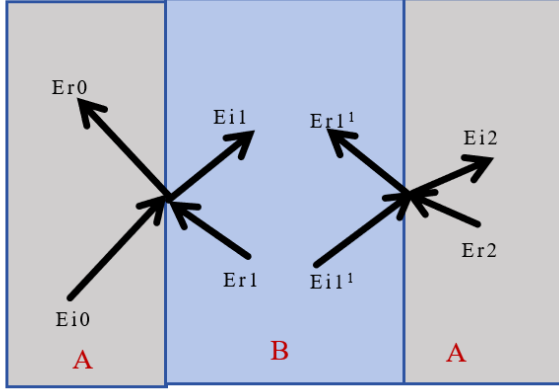


Fig. 1 Schematic diagram of three-layer photonic crystal propagation

$$\begin{cases} k_{si} \sin \theta_i = k_{st} \sin \theta_t \\ k_{qi} \sin \zeta_i = k_{qt} \sin \zeta_t \end{cases}$$

$$\begin{cases} N_{st} = \frac{k_{st}}{k_0} = \sqrt{\frac{1}{2} \left\{ \sqrt{(n_1^2 - k^2 + N_{qi}^2 \sin^2 \zeta_i - N_{si}^2 \sin^2 \theta_i)^2 + 4(n_1 k - N_{si} N_{qi} \sin \theta_i \sin \zeta_i)^2} \right.} \\ \left. + (n_1^2 - k^2 + N_{qi}^2 \sin^2 \zeta_i + N_{si}^2 \sin^2 \theta_i) \right\}} \\ N_{qt} = \frac{k_{qt}}{k_0} = \sqrt{\frac{1}{2} \left\{ \sqrt{(n_1^2 - k^2 + N_{qi}^2 \sin^2 \zeta_i - N_{si}^2 \sin^2 \theta_i)^2 + 4(n_1 k - N_{si} N_{qi} \sin \theta_i \sin \zeta_i)^2} \right.} \\ \left. + (k^2 - n_1^2 + N_{qi}^2 \sin^2 \zeta_i + N_{si}^2 \sin^2 \theta_i) \right\}} \end{cases} \quad (3)$$

N_{si} and N_{qi} in unit layer therefore could be calculated.

Assuming harmonic time dependence $e^{-i\omega t}$, the electric and magnetic fields in complex media take the form:

$$\begin{cases} E(r) = E_0 e^{-i(k_s \vec{s} + i k_q \vec{q})r} \\ H(r) = H_0 e^{-i(k_s \vec{s} + i k_q \vec{q})r} \end{cases} \quad (4)$$

Substituting into Maxwell's equations gives the constitutive relations:

$$\begin{aligned} & \begin{bmatrix} 1 & 1 \\ \frac{1}{\mu_0} (N_{si0} \cos \theta_{i0} + i N_{qi0} \cos \zeta_{i0}) & -\frac{1}{\mu_0} (N_{sr0} \cos \theta_{r0} + i N_{qr0} \cos \zeta_{r0}) \end{bmatrix} \begin{bmatrix} E_{i0,x} \\ E_{r0,x} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ \frac{1}{\mu_1} (N_{si1} \cos \theta_{i1} + i N_{qi1} \cos \zeta_{i1}) & -\frac{1}{\mu_1} (N_{sr1} \cos \theta_{r1} + i N_{qr1} \cos \zeta_{r1}) \end{bmatrix} \begin{bmatrix} E_{i1,x} \\ E_{r1,x} \end{bmatrix} \end{aligned}$$

Matrices M_1, M_2, M_3, M_4 are defined accordingly for both interfaces.

$$\begin{aligned} M1 &= \begin{bmatrix} 1 & 1 \\ \frac{1}{\mu_0} (N_{si0} \cos \theta_0 + i N_{qi0} \cos \zeta_0) & -\frac{1}{\mu_0} (N_{sr0} \cos \theta_0 + i N_{qr0} \cos \zeta_0) \end{bmatrix} \\ M2 &= M3 = \begin{bmatrix} 1 & 1 \\ \frac{1}{\mu_1} (N_{si1} \cos \theta_1 + i N_{qi1} \cos \zeta_1) & -\frac{1}{\mu_1} (N_{sr1} \cos \theta_1 + i N_{qr1} \cos \zeta_1) \end{bmatrix} \end{aligned} \quad (6)$$

In a medium with a complex refractive index, due to the separation of the planes of constant amplitude and phase, their included angle is indeterminate. Therefore, by applying Snell's law separately to the unit vectors of the planes of constant amplitude and phase, the expressions for k_{qt} and k_{st} in the second layer can be obtained, and subsequently, N_{qt} and N_{st} in the second layer are derived, as concluded by Q. Zhang [7].

$$M4 = \begin{bmatrix} 1 & 1 \\ \frac{1}{\mu_0} (N_{si2} \cos \theta_2 + iN_{qi2} \cos \zeta_2) & -\frac{1}{\mu_0} (N_{sr2} \cos \theta_2 + iN_{qr2} \cos \zeta_2) \end{bmatrix}$$

The phase accumulation matrices for each layer incorporate both phase and amplitude contributions:

$$G_1 = \begin{bmatrix} e^{ik_0(N_{sio} \cos \theta_0 + iN_{qio} \cos \zeta_0) da/2} & 0 \\ 0 & e^{-ik_0(N_{sio} \cos \theta_0 + iN_{qio} \cos \zeta_0) da/2} \end{bmatrix}$$

$$G_2 = \begin{bmatrix} e^{ik_0(N_{si1} \cos \theta_1 + iN_{qi1} \cos \zeta_1) db} & 0 \\ 0 & e^{-ik_0(N_{si1} \cos \theta_1 + iN_{qi1} \cos \zeta_1) db} \end{bmatrix} \quad (7)$$

$$G_3 = \begin{bmatrix} e^{ik_0(N_{si2} \cos \theta_2 + iN_{qi2} \cos \zeta_2) da/2} & 0 \\ 0 & e^{-ik_0(N_{si2} \cos \theta_2 + iN_{qi2} \cos \zeta_2) da/2} \end{bmatrix}$$

The total transfer matrix for one period is:

$$M = G_3 M_4^{-1} M_3 G_2 M_2^{-1} M_1 G_1 = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (8)$$

Applying Bloch's condition $e^{iK\Lambda} \begin{bmatrix} E_{i0,x} \\ E_{r0,x} \end{bmatrix} = M \begin{bmatrix} E_{i0,x} \\ E_{r0,x} \end{bmatrix}$ leads to the dispersion relation:

$$e^{iKD} = 1/2 * \left[(m_{11} + m_{22}) \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}m_{21}} \right]$$

The Bloch wave number K is then obtained as:

$$K = -i/D * \ln \left\{ 1/2 * \left[(m_{11} + m_{22}) \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}m_{21}} \right] \right\} \quad (9)$$

For TM polarization, a similar procedure is followed with the magnetic field components, replacing μ with ϵ in the matrix coefficients.

3. Equivalent Transformation for COMSOL Implementation

Direct simulation of lossy periodic structures in COMSOL with full complex wave vector decomposition is non-trivial. However, the transfer matrix derived above can be algebraically manipulated into a form identical to that of a lossless system by defining effective optical parameters.

Taking A-B-A medium as an example, the normal incidence lossless transmission matrix is as follows

For TE polarization

$$Ma = \begin{bmatrix} 1 & 1 \\ na & -na \end{bmatrix}$$

$$Mb = \begin{bmatrix} 1 & 1 \\ nb & -nb \end{bmatrix}$$

The phase accumulation matrices for each layer

$$Ga = \begin{bmatrix} e^{ik_a d_a/2} & 0 \\ 0 & e^{-ik_a d_a/2} \end{bmatrix}$$

$$Gb = \begin{bmatrix} e^{ik_b d_b} & 0 \\ 0 & e^{-ik_b d_b} \end{bmatrix}$$

$$\begin{cases} na' = na^2 / (N_{sio} \cos(\theta_0) + iN_{qio} \cos(\zeta_0)) \\ nb' = nb^2 / (N_{si1} \cos(\theta_1) + iN_{qi1} \cos(\zeta_1)) \\ nc' = na^2 / (N_{si2} \cos(\theta_2) + iN_{qi2} \cos(\zeta_2)) \end{cases} \begin{cases} da' = da/na * (N_{sio} \cos \theta_{i0} + iN_{qio} \cos \zeta_{i0})^2 \\ db' = db/nb * (N_{si1} \cos \theta_{i1} + iN_{qi1} \cos \zeta_{i1})^2 \\ dc' = da/na * (N_{si2} \cos \theta_{i2} + iN_{qi2} \cos \zeta_{i2})^2 \end{cases} \quad (10)$$

With these transformations, the transmission matrix reduces to:

$$T = G_3 M_4^{-1} M_3 G_2 M_2^{-1} M_1 G_1$$

which is formally equivalent to the lossless case. This allows COMSOL users to model lossy photonic crystals by simply replacing the physical refractive index and thickness with the above complex effective values.

For oblique incidence, TE polarization

$$Ma = \begin{bmatrix} 1 & 1 \\ nac \cos(\theta_1) & -nac \cos(\theta_1) \end{bmatrix}$$

$$Mb = \begin{bmatrix} 1 & 1 \\ nbc \cos(\theta_2) & -nbc \cos(\theta_2) \end{bmatrix}$$

The phase accumulation matrices for each layer

$$Ga = \begin{bmatrix} e^{ik_a d_a/2 * \cos(\theta_1)} & 0 \\ 0 & e^{-ik_a d_a/2 * \cos(\theta_1)} \end{bmatrix}$$

$$Gb = \begin{bmatrix} e^{ik_b d_b * \cos(\theta_2)} & 0 \\ 0 & e^{-ik_b d_b * \cos(\theta_2)} \end{bmatrix}$$

For TE polarization, by comparing the matrix elements with the standard lossless TMM, we identify:

$$\begin{cases} na' = na * \cos(\theta_1) \\ nb' = nb * \cos(\theta_2) \end{cases}$$

For lossy periodic structure under oblique incidence by comparing the Equations (6) and (7) with the standard lossless TMM, we identify:

$$\begin{cases} na' = N_{sio} \cos(\theta_0) + iN_{qio} \cos(\zeta_0) \\ nb' = N_{si1} \cos(\theta_1) + iN_{qi1} \cos(\zeta_1) \\ nc' = N_{si2} \cos(\theta_2) + iN_{qi2} \cos(\zeta_2) \end{cases} \quad (10)$$

For TM polarization, the effective index and thickness transformations are:

4. Numerical Validation

To validate the proposed method, we consider a periodic structure with parameters: $n_0 = 2.25$, $n_1 = 1.46 + 0.01i$, $\Lambda = 300nm$, $d_a = \Lambda/2$ and incident angle $\theta = 15^\circ$ under TE polarization. The complex band structures (both real and imaginary parts) obtained from the analytical model exhibit excellent agreement with full-wave COMSOL simulations

across the entire frequency range. Moreover, the real part of the dispersion relation correlates closely with the transmission spectrum, correctly locating the photonic band gaps. By leveraging the closed-form expression in Eq. (9), the

computational time for evaluating the band structure is reduced by nearly half compared to direct lossy-medium simulations, demonstrating a significant improvement in efficiency.

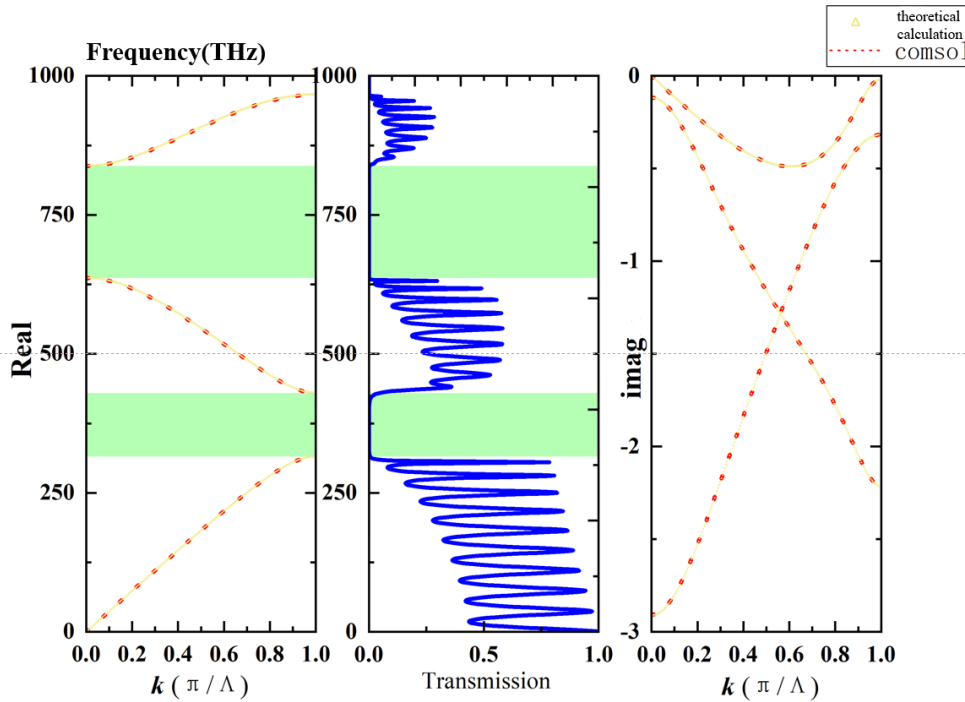


Fig. 2 Band structure and transmission calculated by theory and COMSOL.

5. Summary

This paper presents a rigorous transfer matrix formalism for light propagation in periodic structures containing complex refractive index media. By explicitly decomposing the wave vector into phase and amplitude components, the model accurately captures the non-orthogonal propagation characteristics inherent to lossy materials. The derived dispersion relations and transmission matrices for TE and TM polarizations provide a complete analytical description of the system.

Furthermore, we demonstrate that the transfer matrix for a lossy layer can be mathematically transformed into an equivalent lossless form through effective complex index and thickness renormalization. This enables straightforward implementation in commercial software such as COMSOL Multiphysics without modification to the native physics interfaces. The proposed method bridges the gap between rigorous analytical theory and practical engineering simulation, facilitating the design and optimization of loss-engineered photonic devices.

Future work will extend this approach to anisotropic and nonlinear absorbing media, as well as experimental validation using fabricated dielectric photonic crystals.

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