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Recent Advances in Stability Analysis State Estimation for Time-Delay Neural Networks

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Abstract: This paper reviews recent advances in stability analysis and state estimation for time-delay neural networks (TDNNs). TDNNs are widely applied in various fields, but inherent time delays degrade their performance and induce instability, making the two topics core research focuses. We systematically summarize mainstream stability analysis methods, focusing on the Lyapunov-Krasovskii (L-K) functional method, its improved construction forms, and auxiliary tools like integral and reciprocally convex inequalities. We also synthesize state estimation research progress, emphasizing the core role of estimation error system stability. Finally, we point out existing key problems and prospect future trends, providing a reference for subsequent research.

Keywords: Time-delay neural networks, state estimation, stability analysis, Lyapunov-Krasovskii functional, linear matrix inequality, reduced conservatism.

1. Introduction

As nonlinear system models simulating the structure and function of biological nerve cells, neural networks possess characteristics such as self-learning, self-adaptation, nonlinear mapping, and parallel processing, along with functions including distributed information storage, optimal computation, and pattern memory. They have become important tools for solving complex engineering and information processing problems, and have been applied in numerous key fields such as image processing, associative memory, pattern recognition, combinatorial optimization, and industrial automation [1-5]. Looking back at their development history, the initial theory of neural networks can be traced back to 1943, when American neurobiologist W.S. McCulloch and mathematician W. Pitts first proposed the neuron threshold element MP model [6]. This model systematically explored the action mechanism, structural characteristics of brain cells, and basic physiological properties of biological neurons through mathematical modeling, laying the theoretical foundation for neural networks. Subsequently, to make up for the defect of the MP model in the neuron synaptic regulation mechanism, Canadian psychologist D.O. Hebb proposed the classic Hebb learning rule in 1949 [7]. This rule dynamically adjusts the synaptic connection weight according to the excitation intensity between neurons, becoming the first systematic neural network learning criterion and improving the core mechanism of early neural networks. Although the research on neural networks fell into a low ebb in the 1960s, achievements such as the Self-Organizing Map (SOM) network proposed by Kohonen and the Adaptive Resonance Theory model by Grossberg still accumulated key technologies for the development of the field [8-10]. It was not until the 1980s, with the breakthrough of large-scale integrated circuit technology, that neural network research ushered in a revival period. The proposal of the Hopfield neural network model realized the interdisciplinary integration of physics and physiology, and creatively introduced the energy function to construct a neurodynamic optimization method [11, 12]. Meanwhile, classic models such as Cohen-Grossberg neural networks, BP learning algorithm, and cellular neural networks emerged one after another, gradually improving the theoretical system of neural networks.

With the development of the times, neural networks have attracted much attention due to their dynamic characteristics that are more suitable for complex system modeling, and the widespread existence of time-delay phenomena has become one of the important factors restricting their performance. In the information transmission process of neural networks, factors such as the physical delay of signal transmission between neurons, the limitation of the conversion speed of amplifiers in hardware implementation, and external environmental interference can all lead to the generation of time delays. On the other hand, there are various types of time delays, such as constant time delay, slowly varying time delay, and fast varying time delay, which further increase the complexity of the system. They not only reduce the dynamic response performance of neural networks, but also trigger unstable phenomena such as system chaos and oscillation, seriously affecting their reliability in practical scenarios [13-16]. Therefore, the analysis and design of time-delay neural networks have become one of the core research directions in the field of neural networks. For time-delay neural networks, stability is a necessary prerequisite to ensure their application, and the complete acquisition of state information is the basis for ensuring stability analysis, optimal control, and performance regulation [17-20]. In practical engineering applications, the operating environment of time-delay neural networks often has complex working conditions such as strong interference and noise pollution, and some state variables cannot be directly obtained due to difficulty in direct measurement or high measurement cost. At the same time, problems such as aging of electronic components and parameter drift can also lead to unknown deviations in system states [21-23]. Against this background, state estimation emerged as the times require. Its core goal is to accurately reconstruct the unknown state of the system through measurable output information and the known model of the

system, providing reliable data support for subsequent stability analysis and control strategy design. The state estimation problem of time-delay neural networks can essentially be reduced to the stability analysis problem of the estimation error system. Its core goal is to design an appropriate state estimator so that the error between the estimated state and the real state of the system can asymptotically converge to zero or an allowable range [24-26].

From a theoretical perspective, the research on state estimation of time-delay neural networks focuses on how to reduce the conservatism of stability criteria by improving the construction of Lyapunov-Krasovskii (L-K) functionals [27-31], so as to overcome the problem of limited application scope caused by ignoring time-delay characteristics or excessive scaling in existing research, which is of great significance for improving the theoretical system of neural networks. From an application perspective, the state estimation of time-delay neural networks can provide theoretical support in fields such as industrial automation, image processing, and robot control, and also provide an optimization direction for the engineering application of neural network models such as memristive neural networks and fuzzy neural networks [32-35], thereby promoting the practical application of neural networks.

In summary, the research on state estimation of time-delay neural networks is not only an urgent need to solve practical engineering problems, but also an important way to improve the theoretical system of neural networks. Carrying out in-depth research in this field has irreplaceable theoretical and practical significance for improving the application reliability of time-delay neural networks and promoting the interdisciplinary integration of related disciplines.

2. Stability Analysis of Time-Delay Neural Networks: A Review of Existing Methods

In recent decades, research related to time-delay neural networks has attracted widespread attention from scholars at home and abroad, and fruitful results have been achieved [37-40], such as state estimation of Markov time-delay neural networks, state estimation of time-delay neural networks, etc. Among them, stability analysis is the premise of research on time-delay neural networks. In the process of carrying out research on state estimation of time-delay neural networks, the primary task is to perform stability analysis on the estimation error system, and the analysis results directly determine the performance of the designed state estimator.

In the presence of time delays, in the field of neural network stability analysis, the linear matrix inequality method is the current mainstream solution tool. Looking back at the development history of stability analysis methods for time-delay systems, the frequency domain method was the core technology adopted in the early stage. Its core idea [41] is to determine the system stability by analyzing the root distribution characteristics of the system characteristic equation. However, due to the inherent defect of high solution difficulty of the frequency domain method, its application scenarios are significantly limited. At present, the time domain method has become the mainstream method for stability analysis of time-delay systems. Based on the system state equation, this method is mainly divided into two branches: the Lyapunov-Razumikhin function method and the Lyapunov-Krasovskii (L-K) functional method. Compared with the Lyapunov-Razumikhin function method, the L-K functional method can fully explore and utilize the time-delay information in the system, and the stability criteria obtained accordingly often have lower conservatism, so it has become an important analysis method for the stability research of time-delay systems. The core idea of this method is: construct an appropriate L-K functional, take its derivative, and then judge the stability of the estimation error system according to the positive and negative characteristics of the time derivative of the functional along the system trajectory.

In the process of stability analysis of neural networks in a time-delay environment, the selection of an appropriate L-K functional construction is the key and difficult point of the entire research process. If the L-K functional is unreasonably constructed, it cannot effectively reduce the conservatism of the stability conditions. Therefore, to achieve the research goal of reducing conservatism, scholars at home and abroad have proposed a variety of improved analysis strategies. In terms of the form of functional construction, non-integral term functionals, single integral term functionals, and double integral term functionals are classic forms commonly used in the early stage. With the in-depth research, the academic community has successively proposed new construction forms such as multiple integral functionals, various augmented vector functionals, delay-product-type (DPT) functionals, and asymmetric functionals. It is worth noting that Reference [42] proposed an L-K functional construction scheme with augmented vectors in a single integral term. The single integral term in this functional no longer only contains the state vector, but integrates an augmented vector composed of various elements such as the instantaneous state vector, the delayed state vector, and the single integral term. Since this functional can integrate more information of state variables and time-delay variables, it can effectively reduce the conservatism of the stability conditions. In addition, other new types of L-K functional forms have emerged in the academic community in recent years, such as the Matrix-refined L-K functional [42], which relaxes the positive definite constraint of the quadratic function matrix in the traditional L-K functional, further improving the flexibility of functional construction. The research in Reference [43] shows that integrating time-delay information into functional construction can significantly reduce system conservatism, and thus the DPT-type L-K functional is proposed. At the same time, on the basis of the research in Reference [44], scholars have also developed integral complementary L-K functionals. In recent years, Reference [45] proposed a switched L-K functional, which allows each time-delay mode to correspond to an independent functional instead of sharing one functional for the entire time-delay interval, relaxing the strict requirements of the traditional L-K method on the functional, thereby reducing the conservatism of the stability criteria. In addition, with the increasing complexity of the construction form of the L-K functional, dependent terms containing quadratic time delays will appear in its derivative, which directly makes it impossible to use the linear matrix inequality method for subsequent analysis and solution. To solve this problem, the relevant scholar Kim proposed the negative definite quadratic function method [45], which can effectively handle the dependent terms containing quadratic time-delay terms in the derivative of the L-K functional, and then convert the stability problem into a form solvable by the linear matrix inequality method. It successfully uses the

information of the quadratic time-delay terms, thus obtaining conclusions with lower conservatism. However, at present, how to reasonably introduce more dependent terms containing quadratic time delays into the derivative of the L-K functional and more fully explore and utilize time-delay information to further reduce the conservatism of the conclusions is still one of the key problems to be solved urgently in this research direction.

It can be seen from the above that obtaining stability criteria with lower conservatism is the core research goal in the field of stability analysis of time-delay neural networks. To achieve this goal, in addition to constructing a better L-K functional, it is also crucial to use new integral inequalities to accurately scale the integral terms in the derivative of the L-K functional, which directly affects the conservatism of the research results. In the development process of integral inequalities, the Park inequality constructed in Reference [47] is a milestone achievement. This inequality is extended from the basic inequality in the stability analysis of continuous linear time-varying delay systems, which significantly reduces the conservatism of relevant research results. Subsequently, Reference [48] further generalized the proof method of the Park inequality and proposed the more versatile Moon inequality. Based on the above two types of inequalities, in the derivation process of integral inequalities, the Moon inequality introduces the idea of free-weighting matrices, further reducing the conservatism of the stability criteria. To obtain better stability criteria, the academic community has continuously explored and proposed a series of integral inequality methods: K. Gu first introduced the Jensen integral inequality into the stability analysis of time-delay systems. In 2013, scholars such as Seuret proposed the Wirtinger-based integral inequality [49], which has better scaling accuracy for the boundary of integral terms than the Jensen integral inequality, helping to reduce conservatism. Subsequently, scholars such as Seuret proposed the Wirtinger-based summation inequality [50] on the basis of the Jensen summation inequality, whose scaling accuracy is significantly improved compared with the Jensen summation inequality. The continuous emergence of various new integral inequalities has stimulated the research enthusiasm of scholars at home and abroad on the stability of linear time-delay systems, and also promoted the continuous improvement and development of integral inequality theory. In recent years, new integral inequality achievements have emerged one after another, such as Bessel-Legendre inequality [51], Auxiliary-function-based inequality [52], Free-matrix-based inequality [53], and Wirtinger-based double integral inequality [54], providing rich theoretical tool support for the research on the stability of time-delay neural networks. It is worth noting that in the process of processing integral terms, the accurate estimation of reciprocally convex combinations is an effective way to reduce the conservatism of stability criteria. The estimation accuracy of reciprocally convex combinations directly determines the conservatism of the final stability criteria. Therefore, constructing efficient and accurate reciprocally convex combination estimation inequalities has become a development direction to reduce the conservatism of time-delay system stability criteria. In 2011, Park et al. first proposed the Reciprocally Convex Inequality (RCI), which provides an effective estimation method for reciprocally convex combinations [54], greatly promoting the development of stability analysis of time-delay systems. Since Park et al. proposed the basic RCI, many scholars have improved and extended it to further improve the estimation accuracy and reduce conservatism. Seuret and Gouaisbaut optimized the basic RCI by introducing relaxed matrix variables, expanding its application scope [55]. Zhang et al. further improved the estimation accuracy of reciprocally convex combinations by improving the inequality structure, and proposed the extended RCI [56, 57]. Zeng et al. combined the characteristics of time-delay systems and proposed the hierarchically improved RCI, making it more suitable for the stability analysis of time-varying delay systems. These improved RCIs have been widely used in the stability analysis of various time-delay systems, effectively reducing the conservatism of relevant stability criteria.

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